On Surfaces Containing a System of Cubics that do not Constitute a Pencil.

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1. It is the object of this paper to classify completely the types of algebraic surfaces which are generated by an algebraic system γ of ∞^1 cubic curves which do not constitute a pencil (so that two generic curves of the given system intersect in $\nu \geq 1$ variable points), and to point out a few salient properties of surfaces of the class that belong to types not already known.

I. The Given Surface is Rational.

2. Since a rational surface possesses no integrals of total differentials of the first kind, the cubics of the given system are contained in a linear system of order ν and dimension $r \geq 2$ of cubic curves.*

3. If the cubics of the linear system to which the given systesm γ belongs are rational, we can represent the given rational surface on a plane in such a way that, to the linear system of cubics corresponds a linear system of curves, of some order m, which have a fundamental $(m-1)$-fold point and $t \leq m-1$ simple fundamental points.† Since the given linear system is of order ν, we have

$$m^2-(m-1)^2-t=\nu,$$

and since $t \leq m-1$, we have $m \leq \nu$.

4. If $\nu=1$, we have $m=1$, so that the curves corresponding to the cubics are the right lines in the plane. To the plane or hyperplane sections of the surface correspond curves which intersect the lines of the plane in three points. Hence, *if* $\nu=1$, *the surface generated by the given system of cubics is the surface of order* 9, *belonging to space of nine dimensions, which is represented parametrically by the cubics in a plane, or it is the projection of such a surface.*

*Humbert, "Sur quelques points de la théorie,".... *Journal de Mathématiques*, Ser. 4, Vol. X (1894), p. 195.

†Guccia, *Rendiconti di Palermo*, Vol. I (1887), p. 139. One notices here that a system of curves of odd order on the given surface can not be represented by a system of conics not having a fundamental point, since the curves representing the plane or hyperplane sections must intersect these curves in an odd number of non-fundamental points.

Conversely, on a surface of this type, the cubics corresponding to the tangents to an algebraic curve in the parametric plane constitute a system γ of the required type. This surface was discussed by P. del Pezzo, in the *Rendiconti di Palermo*, Vol. I (1887), p. 241.

5. *If* $\nu>1$, *and if the cubics of the linear system to which the system* γ *belongs are rational, the surface is a ruled surface of order not greater than* 4. In fact, if $\nu=2$, then $m=2$ so that the curves corresponding to the cubics of the linear system are conics through two fixed points P_1 and P_2. Let the curves corresponding to the plane sections be of order n and of multiplicity σ_1 and σ_2 at P_1 and P_2, respectively. Then

$$2n-\sigma_1-\sigma_2=3.$$

Since $\sigma_1+\sigma_2\leq n$, we have $n\leq 3$. It is no restriction to suppose that $n=3$, since the case $n=2$ can be transformed birationally into this one. Then $n=3$, $\sigma_1=2$, $\sigma_2=1$. The surface defined parametrically by such a system of curves is a ruled quartic, belonging to a space of five dimensions, or it is the projection of such a surface.

If $\nu=3$, we find by similar reasoning that the surface is a ruled cubic, belonging to a space of four dimensions, or it is the projection of such a surface.

Finally, if $\nu\geq 4$, the surface is a quadric or a plane.*

6. If a generic cubic of the linear system to which γ belongs is not rational, then the surface is either a cubic surface or a plane. For, if the order of the rational surface containing the system γ exceeds 4, then, by the reference just cited, $\nu=1$ and the cubics of the linear system are rational since their points can be put in (1,1) correspondence with the curves of a linear pencil in the linear system. If the order of the surface equals 4, and the cubics of the linear system are plane cubics, the residual sections of their planes constitute a pencil of right lines on the surface. The cubics are thus rational, since their points are in (1,1) correspondence with the generators of a rational ruled surface.

II. THE SURFACE IS NON-RATIONAL AND OF ORDER NOT GREATER THAN 5.

7. Since the given surface is not rational, a generic cubic of γ is of genus 1† and lies in a plane. If the given surface is of order 3, it is a non-singular cone and the system γ is formed by the sections of the cone by an algebraic system of ∞^1 planes. If the surface is of order 4, the residual sections by the

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* Cf. the author, "Nouvelles Annales de Mathématiques," Ser. 4, Vol. XVII, 1918.

† Castelnuovo–Enriques, "Sopra alcune questioni,".... *Annali di Matematica*, Ser. 3, Vol. VI (1901), p. 213.

planes of the cubics are right lines. The surface is thus a ruled quartic of genus 1, and the system γ is cut from it by an algebraic system of ∞^1 tangent planes.

8. If the given surface is a quintic, the residual section by a plane or hyperplane containing a cubic of γ is a conic. If a generic residual conic is not composite, the surface is generated by a non-rational pencil of conics. Such a surface* has three concurrent double lines and a tacnode. The cubics γ lie in the planes which pass through the tacnode and are tangent to the surface. This surface is normal in three dimensions.

If a generic residual conic is composite, the quintic is a ruled surface of genus 1 belonging to a space of four dimensions or it is the projection of such a surface. The cubic curves on a given ruled quintic surface of genus 1 belonging to S_4 constitute the residual intersections of the surface with the S_3 defined by two generators. Since these cubics constitute a system ∞^1, and intersect in one variable point, they constitute a system γ.

9. *No ruled surface that contains a system γ of cubics is of order greater than* 5. For the generators of such a surface set up, between the points of two generic cubics of γ, a $(1,1)$ correspondence in which at least one point, common to the two cubics, is self-corresponding since it is not multiple on the surface. The order of the surface defined by such a correspondence does not exceed 5.

10. In the remaining cases, we shall transform the given surface birationally into a ruled quintic surface ϕ of genus one, belonging to S_4. We point out here, for use in this connection, some properties of such surfaces ϕ.

11. *Precisely* $\frac{5x(x+1)}{2}$ *independent linear conditions must be satisfied by the coefficients in the equation of an hypersurface* H_1^x *of order* x, *in* S_4, *in order that it contain a given surface* ϕ.

This theorem is true for $x=1$, since ϕ does not lie in an S_3. We assume it true for all orders less than the given order x. Since the rectilinear generators of ϕ intersect each cubic of γ, a necessary and sufficient condition that H^x contains ϕ is that it contains $x+1$ generic cubics $C_1, C_2, \ldots, C_{x+1}$ of γ. An H^x contains the elliptic cubic C_1 if it contains $3x$ generic points on C_1. It then contains C_2 if it contains $3x-1$ generic points on C_2, etc.

These $3x+3x-1+\ldots+2x=\frac{5x(x+1)}{2}$ linear conditions on the coefficients in the equation of an H^x are independent. For, there exists an H^x which

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*Cf. the author, American Journal of Mathematics, Vol. XXX (1908), pp. 115–116.

contains $C_1, C_2, \ldots, C_{j-1}$ (where j has any of the values $1, 2, \ldots, x+1$) and which also contains $3x-j$ generic points $P_j, P_{j+1}, \ldots, P_{3x-1}$ on C_j but which does not contain C_j. There exists, in fact, an hypersurface H_1^{x-1}, of order $x-1$, which contains $C_1, C_2, \ldots, C_{j-2}$ and $P_j, P_{j+1}, \ldots, P_{3x-4}, P_{3x-3}$ and an hyperplane H_1' that contains C_{j-1} and P_{3x-2}. Similarly, there exists a second hypersurface H_2^{x-1}, that contains $C_1, C_2, \ldots, C_{j-2}$ and $P_j, P_{j+1}, \ldots, P_{3x-4}$, P_{3x-2} and a second hyperplane H_2' that contains C_{j-1} and P_{3x-3}. Since the points P are generic, the composite hypersurface $H_1^{x-1}H_1'$ intersects C_j in two points that do not lie on $H_2^{x-1}H_2'$. Thus, no hypersurface of the pencil $\lambda_1 H_1^{x-1}H_1' + \lambda_2 H_2^{x-1}H_2' = 0$ contains C_j. All the hypersurfaces of this pencil contain $C_1, C_2, \ldots, C_{j-1}$ and $P_j, P_{j+1}, \ldots, P_{3x-2}$. One hypersurface of the pencil also contains P_{3x-1}.

12. Any curve on ϕ that intersects a generic generator in x points and a generic cubic of γ in y points is of order $y+2x$, since it intersects the S_3 containing two generic generators and a cubic of γ in $y+2x$ points.

13. *For no curve on ϕ can x be greater than $2y$.* For, the order of such a curve would be $n=y+2x<\frac{5x}{2}$. Since $\frac{5x(x-1)}{2}-n(x-1)>0$, such a curve would lie on an H^{x-1} which does not contain ϕ but has x points in common with each generator.

14. Any curve on ϕ for which $2y=x$ lies on an H^x, since

$$\frac{5x(x+1)}{2}-\frac{5x}{2}\cdot x>0.$$

The residual intersection consists of $5x-\frac{5x}{2}=\frac{5x}{2}$ generators of ϕ, since it does not intersect a generic generator.

There are three curves on ϕ for which $x=2, y=1$. For, put the cubics of γ into $(1,1)$ correspondence with the points of a plane cubic curve C. Then each point of ϕ corresponds to the pair of points on C defined by the two cubics of γ through P. Then the points of a rectilinear generator on ϕ correspond to the pairs of a linear series g_2^1 on C and the points of a cubic of γ to the pairs for which one point is fixed. There are three irrational involutions of order 2 on C defined by the three cubics of which C is the Hessian-Steinerian. Two pairs of such an involution belong to a given g_2^1 and one pair contains a given point P. The points on ϕ defined by these three involutions thus constitute three quintic curves, D_1, D_2, D_3 of genus 1, for which $x=2, y=1$.

We shall show (Art. 35) that there is a rational pencil of curves on ϕ for which $x=4, y=2$. Each of the curves D_1, D_2, D_3 counted twice is a curve of the

pencil. Every curve on ϕ for which $x=2y$ degenerates into curves of this pencil together, possibly, with one or more of the curves D. For let C be such a curve. If C is composite, each component intersects the generators in twice as many points as it does the cubics of γ so that C lies on a proper or composite H^x and forms, with $\frac{5x}{2}$ generators, the complete intersection of H^x with ϕ. Let P be a generic point on C and let C' be the curve of order 10 of the above pencil through P. Then C' has in common with H^x the point P and $10x$ points on the generators common to ϕ and H^x. Hence C' (or a component of it, if P lies on a quintic D) lies on H^x. It forms a component of C since, otherwise, H^x would have more than x points in common with a generic generator of ϕ.

III. The Surface is Non-rational and of Order not Less than 6.

15. Since the order of the surface exceeds 5, the surface is not ruled (Art. 9). The system γ is of order unity (Art. 5, footnote) and index 2.* It is thus representable by pairs of points on a plane cubic curve (cf. Art. 14) and is of genus $p_g=0$, $p_n=-1$.

16. Let the given surface F be projected, if necessary, into a surface F', of order m, in S_3. Then the cubics on F' that constitute the projections of the cubics of γ on F intersect the adjoints to F' of order $m-3$ in just two points which are not multiple on F'. Suppose, in fact, that the cubics do not have a fixed point in common. Since consecutive cubics intersect, the plane of a generic cubic C touches F' at only two of the intersections of C with the plane of the consecutive cubic. The remaining $3m-11$ points common to C and the residual section of its plane with F' lie on the multiple curve and thus on the adjoints of order $m-3$. Similarly, if the cubics have a fixed point P in common, the plane of C touches F' at one point on C. The remaining intersections of C with the residual curve, except one at P, are common to the adjoints of order $m-3$.

17. *The adjoints of order $m-3$ constitute a linear system of at most two dimensions.* Suppose, in fact, that they constituted a linear system of $r>2$ dimensions. Since the cubics are not rational, they determine a linear series g_2^1 on each cubic of γ. Then the ∞^{r-3} adjoints of order $m-3$ through two generic points of one cubic and one generic point of a second generic cubic would contain all the cubics and would thus contain F'. But this is impossible, since the order of the adjoints is less than that of F'.

* Castelnuovo-Enriques, *Mathematische Annalen*, Vol. XLVIII, p. 314.

18. *The genus of the plane sections of F' does not exceed* 4. Let Π be the genus of a generic plane section of F' and let r be the dimension of the linear system of adjoints of order $m-3$. Then*

$$r=\Pi-1-(p_g-p_n) \quad \text{or} \quad \Pi=r+1+p_g-p_n.$$

But $r\leq 2$, $p_g=0$, $p_n=-1$, so that $\Pi\leq 4$. Moreover, $\Pi\geq 3$, since the surface is not ruled.†

19. If $\Pi=3$, it is known‡ that the surface contains an irrational pencil of conics. The quartics cut from such a surface F' by the pencil of adjoints of order $m-3$ degenerate into pairs of conics.

If $\Pi=4$, then $r=2$ so that a pencil of adjoints of order $m-3$ pass through a generic point P on F'. All the adjoints of this pencil pass through two points P_1 and P_2 fixed by P on the cubics C_1 and C_2, respectively, through P. In case P_1 and P_2 lie on a cubic C of γ, they are corresponding points in the g_2^1 defined on C by the adjoints. Then the adjoint surface that contains C contains all the pairs of the g_2^1 on C and thus contains a rational cubic curve (locus of P) which forms, with C, the variable sextic of intersection of the adjoint with F'. In case P_1 and P_2 do not lie on the same cubic, let C_1' and C_2' be cubics of γ through P_1 and P_2, respectively. The pencil of adjoints through P_1 intersect C_1' in P_1 and in a second fixed point distinct from its intersection with C_2. It follows that the adjoint surface that contains C_2 has C_1' for its residual intersection and, similarly, that the one containing C_1 has C_2' for its residual intersection.

20. Let the system γ of cubics on a surface F, belonging to three or more dimensions, be put on $(1,1)$ correspondence with the system of cubic curves on a ruled quintic surface ϕ, of genus 1, belonging to S_4. Then a $(1,1)$ correspondence is set up between F and ϕ by taking two points as corresponding when they lie at the intersection of corresponding pairs of cubics.

21. Let $x>1$ be the order of the rational curves on F. Then the plane or hyperplane sections of F transform into a linear system of curves on ϕ that are of order $3+2x$ since they intersect each cubic in three points and each generator in x points and thus intersect any S_3 which contains a cubic and two generators in $3+2x$ points. Every such curve lies on an H^x, since $\dfrac{5x(x+1)}{2}-x(2x+3)>0$ when $x>1$. The residual intersection of the H^x with ϕ consists of $3(x-1)$ generators. The plane or hyperplane sections of F thus correspond to sections of ϕ by a linear system of H^x through $3(x-1)$ fixed generators.

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* Picard et Simart, "Théorie des Fonctions Algébriques," Vol. II, p. 489.

† Castelnuovo-Enriques, *Mathematische Annalen*, Vol. XLVIII (1897), p. 308.

‡ Scorza, *Annali di Matematica*, Ser. 3, Vol. XVI, p. 255, *et seq.*

The order of the rational curves on F does not exceed 4. For, since $y=3$, we have at once $x\leq 5$ (Arts. 13, 14). But the H^5 through twelve generators define on ϕ a linear system of only two dimensions so that $x\leq 4$.

a. *The Rational Curves on F are Conics.*

22. Since $x=2$ and $y=3$, the curves on ϕ corresponding to the plane or hyperplane sections of F are of order 7 and constitute the residual intersection of ϕ with a system of H^2 through three fixed lines. The complete linear system to which they belong is of order 8 and dimension 5. Hence the surface F is of order 8 and belongs to a space of five dimensions, or it is the projection of such a surface. This surface is the "first type" discussed by Scorza in an article entitled "Le superficie a curve sezioni di genere 3" in the *Annali di Matematica*, Ser. 3, Vol. XVII (1910), p. 320.

b. *The Rational Curves on F are Cubics.*

23. To the system of plane or hyperplane sections of F corresponds a linear system of curves of order 9 cut from ϕ by H^3 through six fixed lines. The complete linear system defined by these curves is of order 9 and dimension 5. Hence, *the given surface F is of order* 9 *and belongs to a space of five dimensions, or it is the projection of such a surface.* A generic hyperplane section is of genus 4 (Arts. 18 and 19).

24. Let F belong to S_5. Since two generic cubics of γ intersect, they lie in an S_4. The residual intersection of this S_4 with F is a cubic of γ since it corresponds to a cubic on ϕ. Since any curve on F intersects three such cubics of γ in the same number of points, we deduce that: *the order of any curve on F is equal to $3y$,* where y is the number of its intersections with a generic curve of γ.

25. There are three types of cubic curves on F, defined by the number x, of their intersections with the rational curves on F. If $x=0$, the curves are the rational cubics; if $x=1$, they are the curves of γ; if $x=2$, there are just three curves D_1', D_2', D_3'. They are of genus 1, and correspond to the three quintics on ϕ (Art. 14) that intersect the generators in two points. The planes of two such cubics D' do not intersect. Otherwise, they would lie in an S_4 which would have four points in common with each rational cubic on F.

26. The rational cubics define a non-rational involution of order 2 on each cubic D'. The lines joining corresponding points of such an involution envelope a curve of class 3 in the plane of D'. It follows that the S_3 defined by the rational cubics on F generate an hypersurface of order 6, since an S_3 which contains a generic line l_1 in the plane of D_1' and l_2, in the plane of D_2', intersects

the locus of the S_3 defined by the rational cubics in a ruled surface which has l_1 and l_2 as triple directrices and is thus of order 6. The point in which the S_3 intersects the plane of D'_3 is a triple point on the sextic. Hence the generator through that point is a triple generator on the ruled surface. It follows that the cubic threespread formed by the lines that intersect the planes of D'_1, D'_2 and D'_3 is a triple threespread on the sextic hypersurface. The hypersurface is of genus 1, and has no other multiple points.

27. Let S be the threespace of a generic rational cubic C on F. The plane of a cubic D' has a line in common with S and determines, with S, an S_4 whose residual intersection with F is a cubic of γ whose plane has a line in common with S. Conversely, if Γ is a cubic of γ whose plane has a line in common with S, then Γ, C and a cubic D' lie in an S_4. Each of the two intersections of Γ with S that do not lie on C lies on a cubic D'. Moreover, the threespread generated by the planes of the cubics γ is of order 9, since it intersects S in three right lines and in the cubic C which is counted twice since two cubics γ pass through each point of C.

28. The S_4 that intersect F in three cubics of γ (Art. 24) define a g^2_3 on the system γ. Let these S_4 be put in (1, 1) correspondence with the points of a plane in such a way that the S_4 corresponding to the points on a line define a g^1_3 on γ. Each cubic of γ belongs to one group of such a g^1_3, so that each point of F, and thus each point of S_5, lies in two of the S_4 define by the points of a line. Then the $\infty^1 S_4$ that pass through a given point correspond to the points of a conic or: the locus of the S_4 that intersect F in three cubics of γ is the dual of a surface of Veronese.

29. The sextics on F intersect the cubics γ in two points (Art. 24). Those that intersect the rational cubics once, correspond to the quartic curves on ϕ. They constitute ∞^1 bundles, intersect in three points and are of genus 1. Three of these bundles constitute the residual intersections of the bundles of S_4 containing a cubic D'. A fourth bundle constitutes the adjoint sextics (Art. 19) to the hyperplane sections of F.

The sextics that intersect the rational cubics twice are of genus 2, intersect in four points and constitute the ∞^1 bundles forming the residual intersections of the S_4 that contain a cubic γ. Those that intersect the rational cubics in three points are of genus 2, intersect in three points and constitute the ∞^1 pencils residual to the rational cubics. Those that intersect the rational cubics in four points are of genus 1 and constitute a pencil (Art. 14).

30. The projection of F on an S_3 from a generic line l in the plane of a cubic C of γ is a sextic surface with a tacnode at the intersection of the plane

of C with S_3. The projections of the cubics of γ pass through the tacnode and lie in pairs in the planes tangent to a quadric cone. The rational cubics also pass through the tacnode. Three of them are nodal, and coplanar with the projections of the cubics D'. The surface has three coplanar double lines, projections of cubics of γ that intersect l. The points of intersection of these double lines are triple points on the surface. The residual nodal cubic passes through these triple points.

31. The projection of F on an S_3 from a generic line l in the plane of a cubic D' is a sextic surface with a triple point at the intersection of the plane of D' with S_3. The rational cubics have a node at this triple point. Their planes envelope a cone of class 3. These planes also contain the cubics γ. The six cubics of γ that intersect l project into double lines forming the six edges of a tetrahedron.

c. *The Rational Curves on F are Quartics.*

32. To the system of plane or hyperplane sections of F corresponds a linear system of curves of order 11 cut from ϕ by H^4 through nine fixed lines. The complete linear system defined on ϕ by these curves is of order 8 and dimension 4. Hence, *the given surface F is of order* 8 *and belongs to four dimensions, or it is the projection of such a surface.* A generic hyperplane section is of genus 4 (Arts. 18 and 19).

33. Let F belong to an S_4. An H^4 that defines a curve on ϕ corresponding to an hyperplane section of F, has eighteen of its twenty intersections with a quintic D (Art. 14) fixed on the nine fundamental lines. These H^4 thus determine a g_2^1 on each quintic D so that *the surface F has three double lines* d_1, d_2, d_3. No two of these double lines can intersect. Otherwise, each S_3 of the pencil containing them would have for residual intersection with F a rational quartic (since it corresponds to a right line on ϕ) from which it would follow that F is rational.

34. The rectilinear generators of ϕ determine on each quintic D a non-rational involution of order 2 which has two pairs of points in common with the g_2^1 defined by the curves corresponding to the hyperplane sections of F. It follows that two rational quartics on F have a double point on each double line.

Let S be the threespace defined by d_2 and d_3. Its residual intersection with F is a rational quartic C_1. The two intersections of C_1 with d_1 coincide at the intersection of d_1 with S. Hence C_1 has a double point on d_1. Similarly, d_1 and d_3 define a quartic C_2 with a double point on d_2 and d_1, d_2 a quartic C_3 with a double point on d_3. The three double points are collinear.

Let C be the rational quartic, other than C_1, that has a double point on d_1. Then, since C does not lie in the S_4 containing d_2 and d_3, C has a double point on a second double line d_2 and lies in a plane. This plane can not contain d_3. Otherwise, the S_3 through it would define a rational pencil of conics on F. Hence the third double point of C lies on d_3.

35. The residual intersections with F of the S_3 through the plane of C constitute a rational pencil of quartics of genus 1 which intersect the rational quartics in four points and the cubics of γ in two points. No two quartics of this pencil intersect. The curves on ϕ corresponding to these quartics are of order 10. They intersect the generators in four points and the cubics of γ twice.

36. No rational quartic on F, other than C, C_1, C_2, C_3, lies in an S_3. Otherwise, let C' be such a quartic. It has a double point on at least two double lines (since it can not lie in the S_3 defined by two double lines) and lies in a plane. The residual intersections of the S_3 containing this plane constitute a pencil of quartics, distinct from that of Art. 35, which intersect the rational curves four times and the cubics of γ twice. This is impossible (Art. 14).

37. Let Γ be any cubic of γ. The plane of Γ intersects a given double line d_1 and defines with it an S_3 whose residual intersection with F is a second cubic Γ' of γ. The points in which d_2 and d_3 intersect S_3 lie on Γ and Γ' and thus on the line of intersection of their planes. It follows that each cubic of γ is intersected in two points (which lie on two of the double lines) by each of three other cubics of γ.

Let S be the threespace defined by d_1 and d_2. Four cubics of γ pass through a generic point of d_1. The lines in which the planes of these cubics intersect S coincide in pairs and intersect d_2. These lines thus establish a $(2, 2)$ correspondence between the points of d_1 and d_2 and generate a ruled quartic surface. The generators of this surface are bisecants of the rational curve C_3 (Art. 34). The common secant line of d_1, d_2, d_3 is a double generator of the ruled quartic.

The hypersurface generated by the planes of γ is of order 8, since its section by the S_3 of two double lines is a ruled quartic counted twice. This octavic hypersurface has F and the three ruled quartic surfaces defined by pairs of the lines d_1, d_2, d_3 as double surfaces. Since it is of genus 1, it has no other multiple points.

38. *The surface F forms, with the plane of the rational plane quartic on it, the basis surface of a pencil of cubic hypersurfaces.* For, the H^3 that contain the three double lines and six generic cubics of γ contain all the cubics of γ and

thus contain F. That an H^3 contain four points on each double line is twelve conditions, that it further contain six generic cubics of γ is twenty-one more conditions on the thirty-five homogeneous coefficients in the equation of the hypersurface. There thus exists a pencil of such H^3. The plane of the rational quartic C (Art. 34) clearly lies on all these H^3 since every line in it has four points in common with each H^3.

39. Let a curve of order n on F intersect a generic rational quartic in x points and a generic cubic of γ in y points. Then

$$4y = x + n.$$

For, the corresponding curve on ϕ is of order $y+2x$. Of its $4(y+2x)$ intersections with the H^4 that defines a generic hyperplane section of F, $9x$ lie on the nine fundamental lines (Art. 32). The remaining $4(y+2x)-9x=n$ define the intersections of the given curve with the given hyperplane. Since $y \leq 2x$, we have $x \leq n$. The equality sign holds (for non-composite curves) only for $x=2$ and $x=4$ (Art. 14).

If $n=2$, we have $x=2$, $y=1$. The curves are the three double lines.

If $n=3$, we have $x=1$, $y=1$. The curves are the cubics of γ.

If $n=4$ and $x=0$, $y=1$, the curves are the rational quartics.

If $n=4$ and $x=4$, $y=2$, the curves are the pencil of residual quartics in the S_3 through the plane of the rational quartic C (Art. 34).

If $n=5$, we have $x=3$, $y=2$. There are ∞^1 pencils of these curves. They form the residual intersections of the S_3 that contain a cubic of γ. They are of genus 2 and intersect in three points.

If $n=6$, we have $x=2$, $y=2$. There are ∞^1 bundles of such curves. They are of genus 2 and intersect in four points. In each pencil in any bundle, there are two which break up into pairs of cubics of γ. One such bundle constitutes the bundle of adjoint sextics to the hyperplane sections of F (Art. 19).

If $n=7$ and $x=1$, $y=2$, we have ∞^1 bundles of curves of genus 1 that intersect in three points. If $x=5$, $y=3$, we have ∞^1 bundles of curves of genus 3 that intersect in five points. All the curves of a bundle intersect each double line in a fixed point.

40. The projection of F on an S_3 from a generic point on a double line d is a sextic surface having a tacnode at the intersection of d with S_3. The tacnodal tangent plane contains the projection of the rational plane quartic and two right lines. The cubics γ lie in pairs in the tangent planes to a quadric cone with vertex at the tacnode. The rational quartics have a double point at the tacnode. The double curve is composed of six right lines forming the edges of a tetrahedron.

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